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1992 J. Phys. A: Math. Gen. 25 4347

(http://iopscience.iop.org/0305-4470/25/16/012)

J. Phys. A: Math. Gen. 25 (1992) 4347–4358. Printed in the UK

Metastable states of the Potts glass

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Received 23 December 1991

Abstract. In this paper we examine the distribution of metastable states in the p-state Potts model. The random-bond Potts model on a chain as well as the uniform bond model on a self-similar lattice are studied. Analytic results are obtained for arbitrary continuous distributions of nearest-neighbour bonds for the chain. When negative bonds are present, the ground state for $p > 2$ is frustrated, in contrast to the Ising model which has $p = 2$. The results for $p > 2$ show some similarity to the discrete random-field Ising model.

1. Introduction

During the past 20 years, spin glasses have been the subject of many theoretical and experimental studies [1–3]. Much of the theoretical effort has been focused on infinite-range models where mean-field theories predict a phase transition independent of the dimension of the lattice. For short-ranged models, the situation is less clear but there seems to be general agreement that the Ising spin glass exhibits a phase transition [4,5] at finite temperature in $d = 3$ but not $d = 2$. This has resulted in other models such as the Potts glass and quadrupolar glasses [1–3, 6, 7] receiving more attention. The dynamics of these random systems is extremely complicated due to the presence of a large number of local energy minima into which the system can be trapped for extremely long times. The distribution of these states as a function of their energy has been studied at zero temperature [8–13] for the Ising model in both $d = 1$ and $d = 2$. In the present paper we study the p-state Potts model on a one-dimensional chain. In particular we study the distribution of metastable states at zero temperature for arbitrary distributions of the bonds between nearest neighbours. Exact results are obtained for all p. Derrida and Gardner [12] have previously studied the Ising model ($p = 2$) for the case of symmetric bond distributions. Their approach is based upon the fact that for continuous distributions the chain has three types of bonds: strong, medium and weak. A weak bond is a bond whose magnitude is smaller than the magnitude of the bonds on either side of it and the number of metastable states is given by 2^w where w is the number of weak bonds. However, in the Potts model with $p > 2$ the sign of the weak bond is important since the spin-inversion symmetry of the Ising model is lost. We extend Derrida and Gardner's results for the distribution of metastable states to arbitrary p and to arbitrary distributions. We also study a uniform-bond model on a self-similar lattice where the fractal geometry is responsible for the disorder. We use a recursive method to calculate numerically the

0305-4470/92/164347+12$04.50

distribution of metastable states in energy. In both cases the distribution of metastable states is similar to that in the random-field Ising model [14, 15]. This latter model is like the p-state Potts model in that the spin-inversion symmetry of the zero-field Ising model is destroyed. In section 2 we describe the model and define a metastable state. The equations for the moments of the distribution of such states are written down and solved analytically. In section 3 we consider the uniform-bond model on a fractal lattice. A recursive algorithm which can be used to obtain the distribution of metastable states numerically is described and the results are discussed. Section 4 summarizes our findings.

2. Metastable states in the Potts glass

The p-state Potts model on a chain of L sites with free ends is defined by the Hamiltonian

$$H = -\sum_{i=1}^{L} J_i(p\delta_{\sigma_i,\sigma_{i-1}} - 1) \tag{1}$$

where the end bonds $J_1 = J_{L+1} = 0$. Each σ_i can take the values $\sigma_i = 1, \ldots, p$ and each J_i is a nearest-neighbour bond which is an independent random variable given by a probability distribution $P(J_i)$. The total energy can be written as

$$E = \sum_{i=1}^{L} E_i(\sigma_{i+1}, \sigma_i, \sigma_{i-1}) \tag{2}$$

where

$$E_i = -\tfrac{1}{2}\left[J_i(p\delta_{\sigma_i,\sigma_{i-1}} - 1) + J_{i+1}(p\delta_{\sigma_i,\sigma_{i+1}} - 1)\right]. \tag{3}$$

If we consider site i and consider 'spin flips' at zero temperature in which σ_i changes while all other σs remain fixed, then there will be a state (or states) for which E_i is minimum for all i. A state is called metastable if such a single 'spin flip' at any site cannot decrease the energy. Such a state may not be stable, however, if two adjacent spins are flipped.

The generating function for the distribution of such single spin-flip states is

$$Z(\lambda) = \sum_{\{\sigma_j\}} \prod_{i=1}^{L} \Theta(-\Delta E_i) e^{-\lambda E_i} \tag{4}$$

where

$$\Theta(-\Delta E_i) = \begin{cases} 1 & \text{if } E_i = \min_{\{\sigma_i\}} E_i(\sigma_{i+1}, \sigma_i, \sigma_{i-1}) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and λ is a Lagrange multiplier that weights the contribution of each state according to its energy. We can rewrite $Z(\lambda)$ as

$$Z(\lambda) = \sum_{\{\sigma_L, \sigma_{L-1}\}} e^{-\lambda E_L(\sigma_{L+1}, \sigma_L, \sigma_{L-1})} \Theta(-\Delta E_L(\sigma_L, \sigma_{L-1})) Y_L(\sigma_L, \sigma_{L-1}) \tag{6}$$

where

$$Y_L(\sigma_L,\sigma_{L-1}) = \sum_{\substack{\{\sigma_j\}\\ 1\leqslant j\leqslant L-2}} \prod_{i=1}^{L-2} \mathrm{e}^{-\lambda E_i(\sigma_{i+1},\sigma_i,\sigma_{i-1})}\Theta(-\Delta E_i(\sigma_{i+1},\sigma_i,\sigma_{i-1})) \tag{7}$$

is a $p \times p$ matrix which can be expressed in the form

$$Y_L(\sigma,\sigma') = A_L\delta_{\sigma,\sigma'} + B_L(1-\delta_{\sigma,\sigma'}) \tag{8}$$

since all p-states are equivalent. For a chain with two sites ($L = 2$), we have the boundary condition that $A_2 = B_2 = 1$, independent of J_2. Hence A_L and B_L can be determined from the following recursive equation:

$$Y_{L+1}(\sigma_{L+1},\sigma_L) = \sum_{\sigma_{L-1}} \mathrm{e}^{-\lambda E_L(\sigma_{L+1},\sigma_L,\sigma_{L-1})}\Theta\left(-\Delta E_L(\sigma_{L+1},\sigma_L,\sigma_{L-1})\right) Y_L(\sigma_L,\sigma_{L-1}) \tag{9}$$

where Y_{L+1} corresponds to the chain with the $(L+1)$th site added.

We first consider the case with $\lambda = 0$ in which all metastable states are weighted equally. Using (7)–(9) we have

$$\begin{pmatrix} A_{L+1} \\ B_{L+1} \end{pmatrix} = \hat{M}(J_{L+1},J_L)\begin{pmatrix} A_L \\ B_L \end{pmatrix} \tag{10}$$

where

$$\hat{M}(J_{L+1},J_L) = \begin{pmatrix} \Theta(J_{L+1}+J_L) & (p-1)\Theta(J_{L+1})\Theta(J_{L+1}-J_L) \\ \Theta(J_L)\Theta(J_L-J_{L+1}) & (p-2)\Theta(-J_L)\Theta(-J_{L+1}) + \Theta(-J_L-J_{L+1}) \end{pmatrix} \tag{11}$$

is a 2×2 transfer matrix which only depends on the last two bonds. This form of $\hat{M}(J_{L+1},J_L)$ is only correct for $p > 2$ since configurations with $\sigma_{i+1},\sigma_i,\sigma_{i-1}$ all different have been included.

For a chain with $L = 3$, A_3 and B_3 are functions of both J_2 and J_3. Averaging these functions over the distribution $P(J_2)$ yields averaged variables $\langle A_3(J_3)\rangle$ and $\langle B_3(J_3)\rangle$. Hence averages for a chain of length $L+1$ can be written in terms of the same quantities for a chain with one less bond as follows:

$$\begin{pmatrix} \langle A_L(J_L)\rangle \\ \langle B_L(J_L)\rangle \end{pmatrix} = \int_{-\infty}^{+\infty} \mathrm{d}J_{L-1} P(J_{L-1})\hat{M}(J_L,J_{L-1})\begin{pmatrix} \langle A_{L-1}(J_{L-1})\rangle \\ \langle B_{L-1}(J_{L-1})\rangle \end{pmatrix}. \tag{12}$$

Assuming in the limit $L \to \infty$ that these variables have the form

$$\begin{pmatrix} \langle A_L(J_L)\rangle \\ \langle B_L(J_L)\rangle \end{pmatrix} = \mu^L \begin{pmatrix} \bar{A}(J_L) \\ \bar{B}(J_L) \end{pmatrix} \tag{13}$$

where μ is the largest eigenvalue of the integral operator (12), we obtain a system of integral equations of the Volterra type for μ. These equations are

$$\bar{A}(J) = \mu^{-1}\left\{\int_{-J}^{\infty} \bar{A}(J')P(J')\mathrm{d}J' + (p-1)\Theta(J)\int_{-\infty}^{J} \bar{B}(J')P(J')\mathrm{d}J'\right\}$$

$$\bar{B}(J) = \mu^{-1}\left\{\Theta(J)\int_{J}^{\infty} \bar{A}(J')P(J')\mathrm{d}J' + \Theta(-J)\int_{0}^{\infty} \bar{A}(J')P(J')\mathrm{d}J' + (p-2)\Theta(-J)\int_{-\infty}^{0} \bar{B}(J')P(J')\mathrm{d}J' + \int_{-\infty}^{-J} \bar{B}(J')P(J')\mathrm{d}J'\right\} \tag{14}$$

where $P(J)$ is the bond distribution and the subscript on J has been dropped.

It is convenient to separate functions of J into positive and negative parts as follows:

$$\begin{aligned}\bar{A}(J) &= A_+(J)\Theta(J) + A_-(-J)\Theta(-J)\\ \bar{B}(J) &= B_+(J)\Theta(J) + B_-(-J)\Theta(-J)\\ P(J) &= P_+(J)\Theta(J) + P_-(-J)\Theta(-J)\,.\end{aligned} \tag{15}$$

For the case of a symmetric bond distribution $P(J) = P(-J)$ we can introduce a new variable

$$\xi = \int_0^J P(J')\mathrm{d}J' \tag{16}$$

and (14) become independent of the detailed form of $P(J)$:

$$\begin{aligned}A_+(\xi) &= \mu^{-1}\left\{\int_0^{1/2} A_+(\eta)\mathrm{d}\eta + \int_0^{\xi} A_-(\eta)\mathrm{d}\eta + (p-1)\int_0^{\xi} B_+(\eta)\mathrm{d}\eta + (p-1)\int_0^{1/2} B_-(\eta)\mathrm{d}\eta\right\}\\ B_+(\xi) &= \mu^{-1}\left\{\int_{\xi}^{1/2} A_+(\eta)\mathrm{d}\eta + \int_{\xi}^{1/2} B_-(\eta)\mathrm{d}\eta\right\}\\ A_-(\xi) &= \mu^{-1}\int_{\xi}^{1/2} A_+(\eta)\mathrm{d}\eta\\ B_-(\xi) &= \mu^{-1}\left\{\int_0^{1/2} A_+(\eta)\mathrm{d}\eta + \int_0^{\xi} B_+(\eta)\mathrm{d}\eta + (p-1)\int_0^{1/2} B_-(\eta)\mathrm{d}\eta\right\}.\end{aligned} \tag{17}$$

A similar set of linear integral equations can be obtained for an arbitrary (non-symmetric) rectangular distribution.

The general solution of this set of equations has the form

$$A_+ = a\mathrm{e}^{\mathrm{i}\alpha\xi} \qquad B_+ = b\mathrm{e}^{\mathrm{i}\alpha\xi} \qquad A_- = c\mathrm{e}^{\mathrm{i}\alpha\xi} \qquad B_- = d\mathrm{e}^{\mathrm{i}\alpha\xi}\,. \tag{18}$$

The parameter $\alpha\mu$ is a solution of the quartic equation

$$(\alpha\mu)^4 - (p+1)(\alpha\mu)^2 + 1 = 0 \tag{19}$$

and the four roots are labelled as $\alpha\mu = \pm y, \pm z$ where

$$y = \{\tfrac{1}{2}[(p+1) + \sqrt{(p-1)(p+3)}]\}^{1/2}$$
$$z = \{\tfrac{1}{2}[(p+1) - \sqrt{(p-1)(p+3)}]\}^{1/2}. \tag{20}$$

In order to have a non-trivial solution of (17), the determinant of the coefficients of (a, b, c, d) must vanish. This condition is

$$\sqrt{(p-1)(p+3)} + (2-y^2)\,[z^2 + (p-2)]\,\frac{\tan(z/2\mu)}{z}$$
$$-(2-z^2)\,[y^2 + (p-2)]\,\frac{\tan(y/2\mu)}{y} = 0. \tag{21}$$

The solution of this equation for μ gives the first moment $\langle N_{ms}\rangle$ of the distribution of metastable states. Derrida and Gardner [12] obtained the result $\mu = 4/\pi$ for $p = 2$. For $p > 2$, the first moment increases with p and in the limit $p \to \infty$ approaches the value $\mu = p/2$.

In addition to the first moment of the distribution of metastable states, we can repeat the calculation described above for arbitrary moments and λ non-zero. A recursive equation for the nth powers of A^n_{L+1} and B^n_{L+1} involve the cross products $A^m_{L+1}B^{n-m}_{L+1}$ with $m = 1, \ldots, n-1$. However these cross products can be written in the form

$$A^m_{L+1}B^{n-m}_{L+1} = \exp\left(\tfrac{1}{2}\lambda m p J_{L+1}\right)\left[C^n_{L+1} + (p-1)^m D^n_{L+1}\right]. \tag{22}$$

The variables C^n_{L+1} and D^n_{L+1} do not depend on m and a set of equations of the following form are obtained:

$$\begin{pmatrix} A^n_{L+1} \\ B^n_{L+1} \\ C^n_{L+1} \\ D^n_{L+1} \end{pmatrix} = \hat{M}_n(J_{L+1}, J_L) \begin{pmatrix} A^n_L \\ B^n_L \\ C^n_L \\ D^n_L \end{pmatrix}. \tag{23}$$

The 4×4 matrix $\hat{M}_n(J_{L+1}, J_L)$ only involves the bonds J_{L+1} and J_L and hence averages of these moments for a chain of length $L+1$ can be expressed in terms of the same quantities for a chain of length L. Introducing averaged variables and separating them into terms for positive and negative bonds, we obtain a set of eight coupled integral equations. There are four simple relations between the eight functions and the set again reduces to four integral equations of the Volterra type.

We first consider the case of $\lambda = 0$ and a symmetric bond distribution $P(J) = P(-J)$. The general solution of these equations is similar to that in (18) and $\alpha\mu$ satisfies the quartic equation

$$(\alpha\mu)^4 - [3p^n - 2(p-1)^n - 1]\,(\alpha\mu)^2 + [2p^n - (p-1)^n - 1]\,[p^n - (p-1)^n]$$
$$-(p-1)^n(2^n - 1)^2 = 0. \tag{24}$$

This equation reduces to (19) for the first moment when $n = 1$. The integral equations have a non-trivial solution when the following condition is satisfied:

$$\bar{y}\bar{z}(\gamma_2 - \gamma_1) - \bar{y}\tan(\bar{z}/2\mu)[1-\gamma_1)(1 + (p-1)^n\gamma_2]$$
$$+ \bar{z}\tan(\bar{y}/2\mu)[1-\gamma_2)(1 + (p-1)^n\gamma_1] = 0 \tag{25}$$

with

$$\gamma_1 = \frac{\bar{y}^2 + (p-1)^n - 2p^n + 1}{(p-1)^n(2^n-1)}$$
$$\gamma_2 = \frac{\bar{z}^2 + (p-1)^n - 2p^n + 1}{(p-1)^n(2^n-1)} \tag{26}$$

where $\bar{y}$ and $\bar{z}$ are the absolute values of the roots of the quartic equation in (24) and reduce to y and z when $n = 1$. If the roots corresponding to $\bar{y}$ or $\bar{z}$ are complex then in (25) the corresponding tan function is replaced by tanh. In the limit of large p and fixed n, we have $\mu = p^n/2$ whereas in the limit of large n and fixed p, $\mu = (p-1)^n/2$.

The average of the logarithm of the number of metastable states can be obtained by examining the limit $n \to 0$ of the nth moment. We give below analytic results for $\Lambda = \lim_{n\to 0} \partial\mu/\partial n$ in the case where the distribution is weighted according to energy using λ as in (4) and for an arbitrary distribution of bonds. The average of the logarithm of the number of metastable states is related to Λ by

$$\langle \ln N_{ms} \rangle = \Lambda + \lambda E \qquad E = -\frac{\partial \Lambda}{\partial \lambda} \tag{27}$$

where E is the average energy of the states. With $P_+(J)$ and $P_-(J)$ defined as in (15) we also define

$$\Phi_\pm(J) = \int_J^\infty P_\pm(J')\mathrm{d}J'$$
$$\langle J \rangle_\pm = \int_0^\infty J' P_\pm(J')\mathrm{d}J' . \tag{28}$$

The solution of the equation which is the generalization of (25) in the limit $n \to 0$ gives

$$\begin{aligned}\Lambda = \Phi_-(0)\ln(p-1) &+ \lambda\left[(p-1)\langle J\rangle_+ + \langle J\rangle_-\right] \\ &+ \int_0^\infty \mathrm{d}J'\{f_1(J')\Phi_+(J')P_-(J')\Phi_+(J') \\ &+ f_1(J')\Phi_-(J')P_+(J')\Phi_-(J') + 2f_2(J')\Phi_+(J')P_+(J')\Phi_-(J') \\ &+ f_3(J')\Phi_+(J')P_+(J')\Phi_+(J')\}\end{aligned} \tag{29}$$

where

$$f_1(J') = \ln\left[1 + \frac{\mathrm{e}^{-\lambda p J'}}{p-1}\right]$$
$$f_2(J') = \ln\left[1 + \mathrm{e}^{-\lambda p J'}\right] \tag{30}$$
$$f_3(J') = \ln\left[1 + (p-1)\mathrm{e}^{-\lambda p J'}\right] .$$

Each of the terms inside the integrand in (29) corresponds to a weak bond configuration but configurations with two adjacent negative bonds do not contribute. This is in contrast to the Ising model ($p = 2$) where the sign of the weak bonds is irrelevant. In the case $p = 2$ all three functions $f_i (i = 1,2,3)$ are the same and the additional weak bond configurations must also be included. Hence for the Ising model we have

$$\begin{aligned}\Lambda = \lambda\left[\langle J\rangle_+ + \langle J\rangle_-\right] + \int_0^\infty \mathrm{d}J' \left\{\ln\left[1 + \mathrm{e}^{-2\lambda J'}\right]\right\} \{&\Phi_+(J')P_-(J')\Phi_+(J') \\ &+ \Phi_-(J')P_+(J')\Phi_-(J') + 2\Phi_+(J')P_+(J')\Phi_-(J') \\ &+ \Phi_+(J')P_+(J')\Phi_+(J') + 2\Phi_-(J')P_-(J')\Phi_+(J') \\ &+ \Phi_-(J')P_-(J')\Phi_-(J')\}\,. \end{aligned} \tag{31}$$

Thus (29) and (31) can be used to study non-symmetric distributions of the bonds.

As $\lambda \to \infty$, only the lowest energy states are included and we have

$$\begin{aligned}\Lambda &= \Phi_-(0)\ln(p-1) \\ \bar{E} &= E_{\min} = -(p-1)\langle J\rangle_+ - \langle J\rangle_-\end{aligned} \tag{32}$$

and thus the ground state is highly degenerate for $p > 2$ if negative bonds are present. The maximum energy $E_{\max}$ and the most probable energy E_{mp} are also easily obtained by taking the limits $\lambda \to -\infty$ and $\lambda \to 0$ respectively. The results for arbitrary non-symmetric distributions are easily obtained from (27)–(31). For example, if all bonds are negative then the distribution collapses to $\langle \ln N_{\mathrm{ms}}\rangle = \ln(p-1)$, $\bar{E} = -\langle J\rangle_-$. If all bonds are positive, then the result is essentially the same as for the case $p = 2$. Figure 1 shows $\langle \ln N_{\mathrm{ms}}\rangle$ as a function of $\bar{E}$ for a Gaussian distribution of bonds with non-zero mean $J_0 = \pm 0.5$ and unit variance in the case $p = 3$. The distribution has discontinuities at $E_{\min}$ and $E_{\max}$ in both cases. These discontinuities are due to the presence of adjacent negative bonds in the chain. As the number of positive bonds increases ($J_0 = +0.5$) these discontinuities decrease with the one at $E_{\min}$ decreasing more rapidly. When all bonds are positive both discontinuities are zero and the curve is symmetric about E_{mp} where it has the value $\frac{1}{3}\ln p$. However as the number of negative bonds increases ($J_0 = -0.5$) the discontinuities increase and the distribution shrinks in width until finally when all bonds are negative it collapses to the value $\ln(p-1)$.

The expressions simplify in the case of a symmetric distribution and we have

$$\Lambda = \tfrac{1}{2}\ln(p-1) + \tfrac{1}{2}\lambda p\langle J\rangle + \int_0^\infty \mathrm{d}J'\{2f_1 + 2f_2 + f_3\}\Phi_+^2(J')P_+(J') \tag{33}$$

where we have used the fact that

$$\Phi_+(0) = \Phi_-(0) = \tfrac{1}{2} \qquad \langle J\rangle_+ = \langle J\rangle_- = \tfrac{1}{2}\langle J\rangle\,. \tag{34}$$

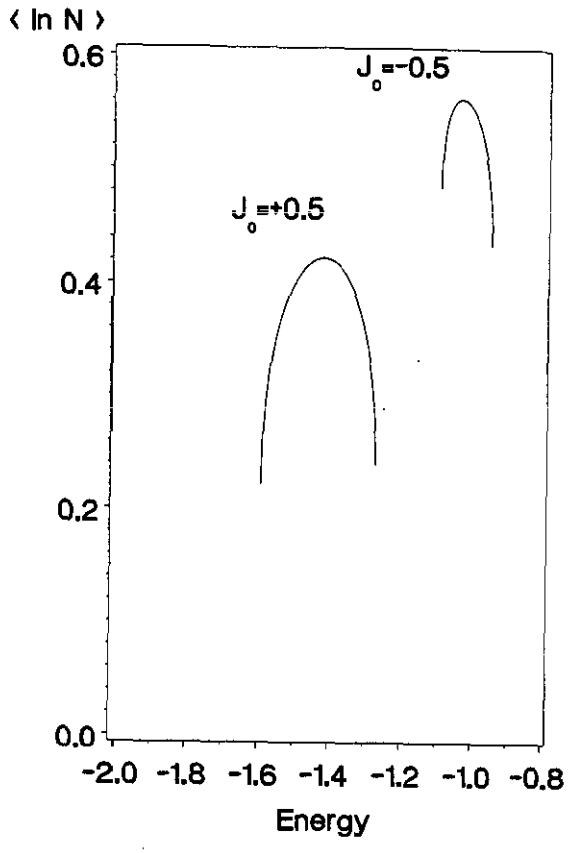

Figure 1. The average of the logarithm $\langle \ln N_{ms} \rangle$ as a function of energy for $p = 3$ with a Gaussian distribution of bonds having unit variance and non-zero mean $J_0 = \pm 0.5$.

In the limits $\lambda \to \infty, 0, -\infty$ respectively, we find

$$\begin{aligned}
&\langle \ln N_{ms} \rangle_{\lambda \to \infty} = \tfrac{1}{2} \ln(p-1) \\
&\bar{E}_{\lambda \to \infty} = E_{\min} = -\frac{p}{2} \langle J \rangle \\
&\langle \ln N_{ms} \rangle_{\lambda \to 0} = \tfrac{1}{24} (10 \ln(p-1) + 3 \ln p + 2 \ln 2) \\
&\bar{E}_{\lambda \to 0} = E_{mp} = E_{\min} + (2p+1) I \\
&\langle \ln N_{ms} \rangle_{\lambda \to -\infty} \simeq \tfrac{11}{24} \ln(p-1) \\
&\bar{E}_{\lambda \to -\infty} = E_{\max} = E_{\min} + 5pI
\end{aligned} \tag{35}$$

where

$$I = \int_0^\infty J \Phi_+^2(J) P_+(J) \mathrm{d}J \tag{36}$$

is simply the average energy of the positive weak bonds. These results are valid for any continuous symmetric distribution $P(J)$ and are plotted in figure 2 in the case

of $p \simeq 3$ for a Gaussian distribution of bonds with zero mean. For $p = 2$, (31) must be used and the value of $\langle \ln N_{ms} \rangle$ is zero at E_{min} and E_{max} and equal to $\frac{1}{3}\ln 2$ at E_{mp}. The coefficient of I in (35) is 8 for E_{mp} and 16 for E_{max} in this case.

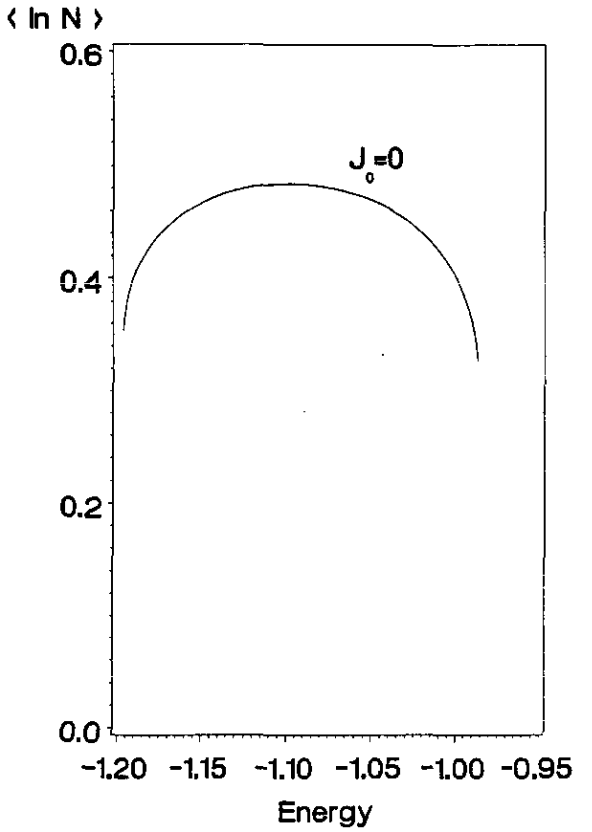

Figure 2. The average of the logarithm $\langle \ln N_{ms} \rangle$ as a function of energy for $p = 3$ with a Gaussian distribution of bonds having unit variance and zero mean.

For the Ising model the distribution of metastable states is symmetric about E_{mp} but for $p > 2$ this is no longer the case. The distribution has discontinuities at E_{min} and E_{max} with that at E_{min} being largest. As mentioned above, these discontinuities are due to the presence of adjacent negative bonds in the chain. This behaviour is very similar to the discrete $\pm h$ random-field Ising model with $h < 2J$ where J is the non-random bond. Masui [14, 15] has studied the distribution of metastable states in this model and found that the distribution is both asymmetric about E_{mp} and has discontinuities at the edges. The degeneracy at the edges is related to the many different ways in which domain walls can form whenever the random field changes sign from one site to the next. However, for the continuous random field model, Masui [14, 15] finds that there are no discontinuities at the edges and that the distribution is much more symmetric about E_{mp}. These results would suggest that the discreteness of the Potts model is important in the present study and that perhaps these degeneracies would disappear in quadrupolar glasses [1–3] where all orientations are permitted.

3. Potts model on a fractal

In the previous section we studied the random-bond Potts model on a regular $d = 1$ chain. The metastable states are due to the competition between unequal bonds. In systems of higher-dimension, metastable states can arise even if the bonds all have the same value. The simplest example of such a system is a self-similar plane lattice with odd coordination number. Fractal lattices provide a useful tool for the study of disorder since exact results for the thermodynamic functions of statistical mechanical models can be obtained using real space renormalization group methods [16]. Although the lattices are rather artificial, they have an effective dimensionality $D > 1$.

In this section we consider the p-state Potts model on the self-similar lattice shown in figure 3. This lattice was first introduced by Nelson and Fisher [16] to study the influence of dimensionality on the critical properties of the Ising model. In our case each vertex has a Potts variable σ_i which can take p values and each solid line connecting the three neighbouring vertices represents a bond of strength J in the Potts model Hamiltonian. Thus all bonds have the same strength and the disorder is due to the positions of the Potts variables. The fractal dimension [16] of this lattice is $D = \ln 3/\ln 2$. The equilibrium partition function can be obtained exactly using real space rescaling methods and the model exhibits long-range order only at zero temperature as in the chain. Bell and Southern [17] have previously studied the dynamics of the $p = 2$ Ising model on this lattice and found that there are an infinity of divergent relaxation times at zero temperature due to the presence of metastable states. All bonds are equal and positive but there are an infinite number of states which are stable against single spin flips at zero temperature. For any value of p, these states correspond to the σ_is in each elementary triangle having the same state. The distribution of these states with respect to energy at zero temperature can be obtained numerically using a recursive method based on (4) in section 2 . The trace over the σ_i on the smallest scale is performed and only metastable states are included. This procedure is repeated again at the next length scale until the logarithm of the number of metastable states converges. The variable λ in (4) is used to weight the contribution of each state according to its energy. There are no weak bonds but a hierarchical structure of metastable states can be identified.

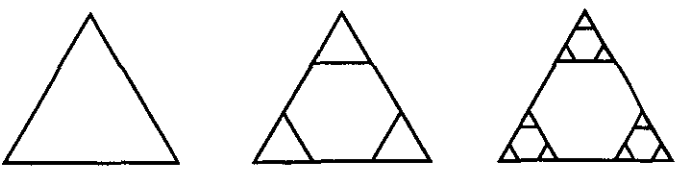

Figure 3. The first three stages of the 3-simplex lattice.

Figure 4 shows the logarithm of the number of states (entropy) as a function of the energy of the states for $p = 3$. The broken curve represents the equilibrium entropy of the Potts model as a function of equilibrium energy and the slope of this curve is the inverse temperature. The full curve represents states that are metastable at zero temperature but which do not have any weight in the equilibrium properties at zero temperature except for those at the minimum energy. For this lattice there is no essential difference between the case with $p = 2$ and $p > 2$. The degeneracy of the metastable states at E_{max} is due to the discreteness of the Potts model and occurs for

the same reason that the anti-ferromagnetic Potts model has a ground-state entropy on this lattice. At finite temperatures, these states do not have infinite relaxation times but they are still long and, as can be seen by comparing the two curves, these states account for a large fraction of the total number of equilibrium states at low temperatures.

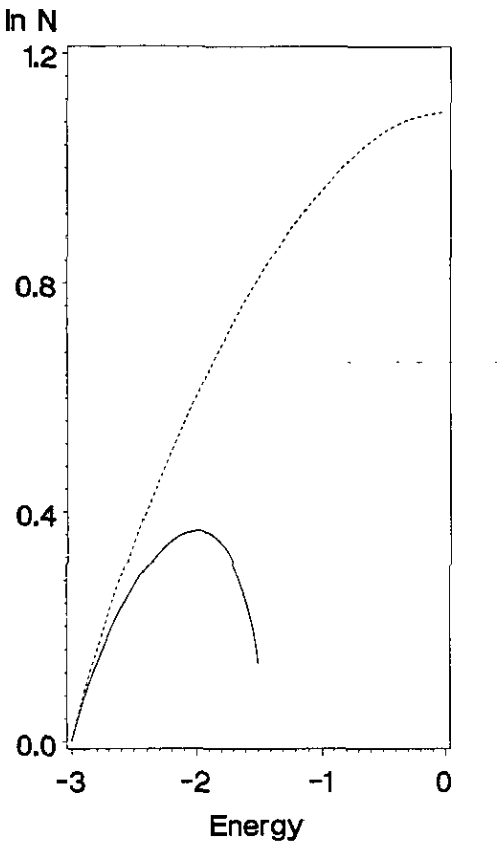

Figure 4. The average of the logarithm of the number of metastable states at zero temperature as a function of energy (full curve) and the logarithm of the number of equilibrium states (broken curve) as a function of the equilibrium energy for the $p = 3$-state Potts model on the 3-simplex lattice.

4. Summary

We have obtained exact results for the distribution of metastable states at zero temperature for the p-state Potts model on a linear chain for arbitrary continuous distributions of the nearest-neighbour bonds. In contrast to the Ising spin glass, there is a large degeneracy due to the presence of adjacent negative bonds on the chain. In the case of symmetric distributions, the distribution of these states as a function of energy is not symmetric about the most probable energy and there are discontinuities at the minimum and maximum energies. We have also studied the uniform-bond

Potts model on a self-similar lattice. In this case the disorder is due to the position of the Potts variables but the results for $p = 2$ and $p > 2$ are essentially the same. The distribution of metastable states as a function of energy is not symmetric about the most probable energy and has a discontinuity at the upper edge.

For both the random-bond chain and the self-similar lattice these metastable states have long but finite relaxation times at non-zero temperatures and they account for a significant fraction of the equilibrium states at low temperatures.

Acknowledgment

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

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